Feynman perturbation expansion for the price of coupon bond options and swaptions in quantum finance. II. Empirical

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The quantum finance pricing formulas for coupon bond options and swaptions derived by Baaquie [Phys. Rev. E **75**, 016703 (2006)] are reviewed. We empirically study the swaption market and propose an efficient computational procedure for analyzing the data. Empirical results of the swaption price, volatility, and swaption correlation are compared with the predictions of quantum finance. The quantum finance model generates the market swaption price to over 90% accuracy.

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I. INTRODUCTION

A swaption is an interest rate option that has a deep and liquid market, and arguably is even the most liquid option on interest rates that is available today. Participants in the swaption market tend to focus their attention on the highly liquid European swaptions, that is, swaptions that can be exercised only on a single date. The theoretical price of these instruments depends on the forward interest rates model being used.

In quantum finance the forward interest rates are modeled as a two-dimensional quantum field theory [2] and has shown great advantage in pricing and hedging interest rate options [3,4]. This encourages us to investigate the swaption market in some detail within the quantum finance framework. Swaptions can be expressed as a special case of coupon bond options. Swaption pricing is a nonlinear problem which usually needs to be studied numerically. However, as shown in Baaquie [1], henceforth called paper I, Feynman perturbation expansion gives a closed form approximation for the price and correlation of swaptions. All the pricing formulas and empirical results for swaptions obtained in this paper are also valid for the coupon bond option.

In Sec. II the relevant formulas for swaption price, volatility, and correlation are reviewed and the computational framework is defined. In Sec. III the data of swaptions and zero coupon yield curve are discussed in some detail. Data processing and computational procedures are discussed in Sec. IV with some empirical results being given in Sec. V.

II. PRICE AND CORRELATION OF SWAPTIONS

We review the results, given in paper I, on swaption price, volatility, and correlation that provide the analytical basis for our empirical study.

Swaps and swaptions are financial instruments that are widely used in the debt market. Swaps are interest rate derivatives in which one party pays floating interest rate, determined by the prevailing Libor at the time of the payment, with the other party paying at a prefixed interest rate R_S. The swaptions that we are studying have floating interest rate payments that are paid at $\ell=3$ month intervals and fixed rate payments that are paid at intervals of $2\ell=6$ months. The three monthly floating rate payments are paid at times $T_0+n\ell$, with $n=1,2,\ldots,N$; there are N payments. For six monthly fixed rate payments there are only $N/2$ payments[1] of amount $2R_S$, made at times $T_0+2n\ell$, $n=1,2,\ldots,N/2$.

The payoff function for the interest rate swaption, in which the holder of the option receives at the fixed rate and pays at the floating rate, is given in paper I by [5]

$$
\begin{aligned}
C(T_0;R_S) &= V\left(B(T_0,T_0+N\ell)+2\ell R_S\sum_{n=1}^{N/2}B(T_0,T_0+2n\ell)-1\right)_+\\
&= V\left(\sum_{n=1}^{N/2}c_nB(T_0,T_0+2n\ell)-1\right)_+, \qquad (1)
\end{aligned}
$$

where $B(t,T)$ is the price of a zero coupon Treasury Bond at time t that matures at time $T>t$. The coefficients and strike price for a swaption are hence given by

$$
c_n = 2\ell R_S, \quad n=1,2,\ldots,(N-1)/2,
$$

$$
\text{payment at time } T_0+2n\ell,
$$

$$
c_{N/2} = 1+2\ell R_S, \quad \text{payment at time } T_0+N\ell,
$$

$$
K=1. \qquad (2)
$$

The fixed interest rate par value R_P, at time t_0, is such that the value of the interest rate swap has zero value. Hence

$$
\sum_{n=1}^{N/2}c_nB(t_0,T_0+2n\ell)-1 = 0 \Rightarrow 2\ell R_P(t_0) = \frac{B(t_0,T_0)-B(t_0,T_0+N\ell)}{\sum_{n=0}^{N/2}B(t_0,T_0+2n\ell)}. \qquad (3)
$$

The price of a coupon bond option $C(t_0,t_*,R_S)$ at time $t_0<t_*$, using the money market measure and discounting the value of the payoff function using the spot interest rate $r(t)=f(t,t)$, is given from Eq. (1) by

[1] Suppose the swaption has a duration such that N is even. Note $N=4$ for a year long swaption.

$$C(t_0,t_*,R_S) = VE\left[e^{-\int_{t_0}^{t_*} dt r(t)} \left(\sum_{n=1}^{N/2} c_n B(t_*,t_* + 2n\ell) - 1 \right)_+ \right], \tag{4}$$

where V is the notional deposit on which the interest is calculated; we set $V=1$.

The option price has been derived in paper I and yields the results given below,[2]

$$C(t_0,t_*,R_S) = B(t_0,t_*)\sqrt{\frac{A}{2\pi}}\left(\frac{B}{6A^{3/2}}X + \frac{C}{24A^2}(X^2-1) + \frac{1}{72}\frac{B^2}{A^3}(X^4 - 6X^2 + 3) \right) e^{-(1/2)X^2} + B(t_0,t_*)\sqrt{\frac{A}{2\pi}} I(X) + O(\sigma^4), \tag{5}$$

$$I(X) = e^{-(1/2)X^2} - \sqrt{\frac{\pi}{2}} X \left[1 - \Phi\left(\frac{X}{\sqrt{2}} \right) \right], \tag{6}$$

$$X = \frac{K-F}{\sqrt{A}}, \tag{7}$$

$$F \equiv \sum_{i=1}^{N} J_i,$$

$$J_i \equiv c_i F_i, \quad F_i \equiv F_i(t_0,t_*,T_i) = \exp\left(-\int_{t_*}^{T_i} dx f(t_0,x) \right) \tag{8}$$

F_i are the forward bond prices; coefficients c_i and strike price K are given in Eq. (2) in terms of the fixed interest rate R_S. For a swaption initialized at time t_0 to be at the money, the fixed interest rate R_S is equal to the par value $R_P(t_0)$.

The coefficients in the option price are given in paper I by

$$A = \sum_{ij=1}^{N} J_i J_j \left(G_{ij} + \frac{1}{2} G_{ij}^2 \right) + O(G_{ij}^3),$$

$$B = 3 \sum_{ijk=1}^{N} J_i J_j J_k G_{ij} G_{jk} + O(G_{ij}^3),$$

$$C = 16 \sum_{ijkl=1}^{N} J_i J_j J_k J_l G_{ij} G_{jk} G_{kl} + O(G_{ij}^4). \tag{9}$$

The market correlator G_{ij} of the forward bond prices has been derived in paper I. G_{ij} for different quantities is defined over different domains of the forward interest rates and this results in the integration of the forward interest rates correlation function over different integration limits. The exact form of the various integrations will be discussed later with the other correlators that are required for the computation of swaption volatility.

The input data that we need for computing the swaption price can be derived from the underlying forward interest rates' data and yield the coupon bond price, the forward bond price, and the fixed rate par value R_P.

A. Swaption at the money

Recall the par value of the fixed interest rate $R_P(t_0)$ is the value for the fixed interest payments for which the swap at time t_0 is zero. From Eqs. (8) and (3) fixed interest rate equal to par value, namely $R_S=R_P$, implies the following:

$$\begin{aligned} F \equiv F(t_0) &= \sum_{i=1}^{N/2} c_i F(t_0,T_0,T_0+2i\ell) \\ &= \sum_{i=1}^{N/2} 2\ell R_P F(t_0,T_0,T_0+2i\ell) + F(t_0,T_0,T_0+N\ell) \\ &= \frac{B(t_0,T_0) - B(t_0,T_0+N\ell)}{\sum_{n=0}^{N/2} B(t_0,T_0+2n\ell)} \sum_{i=1}^{N/2} F(t_0,T_0,T_0+2i\ell) \\ &\quad + F(t_0,T_0,T_0+N\ell) = 1. \end{aligned} \tag{10}$$

In the coupon bond option pricing formula $X=(F-K)/\sqrt{A}$ and for swaptions $K=1$. Hence when the fixed interest rate R_S for the swaption is at the money $F=1$ and this leads to $X=(F-K)/\sqrt{A}=0$. As discussed in paper I, the asymptotic behavior of the error function yields the following:

$$I(X) = 1 - \sqrt{\frac{\pi}{2}} X + 0(X^2), \quad X \approx 0 \tag{11}$$

and hence the swaption close to the money, to leading order, has the form

$$C(t_0,t_*,R_P) \simeq B(t_0,t_*)\sqrt{\frac{A}{2\pi}} - \frac{1}{2} B(t_0,t_*)(K-F) + O(X^2). \tag{12}$$

B. Volatility and correlation of swaptions

The volatility and correlation of swaption prices are important quantities since they are indicators of the market's direction and also give us insights on portfolio study.

Consider the volatility and correlation of the change of swaption price for infinitesimal time steps. Let $C_1 \equiv C(t_0,t_1,R_1)$ and $C_2 \equiv C(t_0,t_2,R_2)$ denote two swaptions. Denote time derivative by an upper dot; for infinitesimal time step ϵ we have

$$\begin{aligned} \langle \dot{C}_1 \dot{C}_2 \rangle_c &= \frac{1}{\epsilon^2} \langle [C_1(t_0+\epsilon) - C_1(t_0)][C_2(t_0+\epsilon) - C_2(t_0)] \rangle_c \\ &= \frac{1}{\epsilon^2} \langle \delta C_1(t_0) \delta C_2(t_0) \rangle_c, \end{aligned} \tag{13}$$

where the connected correlator is defined by $\langle AB \rangle_c \equiv \langle AB \rangle - \langle A \rangle \langle B \rangle$.

[2]The error function is given by $\Phi(u)=(2/\sqrt{\pi})\int_0^u dW e^{-W^2}$.

Note the swaption prices C_1, C_2 depend on the forward bond prices F_i, which take random values every day. The random changes in the price of the forward bond prices lead to changes in the price of the swaption. The correlation function $\langle \delta C_1(t_0) \delta C_2(t_0) \rangle_c$ can be evaluated by a historical average over the daily swaption prices, considered as the random outcomes of the swaption price due to the random changes in the forward bond price. Hence a historical average of the correlator of changes in the swaption price can be equated to the ensemble average of the correlator taken over the random fluctuations of the forward bond prices.

The field theory of forward interest rates yields, from paper I, that

$$\langle \dot{f}(t,x) \dot{f}(t,x') \rangle_c = \frac{1}{\epsilon} M(t,x,x'). \tag{14}$$

From the pricing formula given in Eq. (12), the swaption's rate of change at the money, namely $X=0$, is given by the following:

$$\begin{aligned}\sqrt{2\pi}\frac{dC(t_0,t_I,R_P)}{dt_0} &= \frac{dB(t_0,t_I)}{dt_0}\sqrt{A_I} + \frac{1}{2\sqrt{A_I}}\frac{dA_I}{dt_0} + \sqrt{\frac{\pi}{2}}B(t_0,t_I)\frac{dF}{dt_0} \\ &= D_I - C(t_0,t_I,R_P)\int_{t_0}^{t_I} dx\dot{f}(t_0,x) - \frac{B(t_0,t_I)}{\sqrt{A_I}}\sum_{ij=1}^{I} J_i J_j G_{ij}\int_{t_I}^{T_j} dx\dot{f}(t_0,x) - \sqrt{\frac{\pi}{2}}B(t_0,t_I)\sum_{i=1}^{N} J_i\int_{t_I}^{T_i} dx\dot{f}(t_0,x),\end{aligned} \tag{15}$$

where t_I denotes t_1 or t_2 and D_I contains all the deterministic (nonstochastic) factors that are subtracted out in forming the connected correlation functions.

To determine $\dot{C}$, as seen from the equation above, one needs $\dot{f}(t_0,x)$, namely the evolution equation of the quantum field $f(t,x)$, and is discussed in paper I. The evolution equation of $f(t,x)$, together with Eqs. (14) and (15), yields the following:

$$\begin{aligned}2\pi\epsilon\langle \delta C_1(t_0)\delta C_2(t_0)\rangle_c &= C_1C_2\int_{t_0}^{t_1} dx\int_{t_0}^{t_2} dx' M(t_0,x,x') + \frac{B(t_0,t_2)}{\sqrt{A_2}}C_1\sum_{jj'=1}^{N2} G_{jj'}J_jJ_{j'}\int_{t_0}^{t_1} dx\int_{t_2}^{T_j} dx' M(t_0,x,x') \\
&+ \frac{B(t_0,t_1)}{\sqrt{A_1}}C_2\sum_{ii'=1}^{N1} G_{ii'}J_iJ_{i'}\int_{t_0}^{t_2} dx\int_{t_1}^{T_i} dx' M(t_0,x,x') \\
&+ \frac{B(t_0,t_1)B(t_0,t_2)}{\sqrt{A_1A_2}}C_1C_2\sum_{ii'=1}^{N1}\sum_{jj'=1}^{N2} G_{ii'}J_iJ_{i'}G_{jj'}J_jJ_{j'}\int_{t_1}^{T_i} dx\int_{t_2}^{T_j} dx' M(t_0,x,x') \\
&+ \sqrt{\frac{\pi}{2}}B(t_0,t_2)C_1\sum_{j=1}^{N2} J_j\int_{t_0}^{t_1} dx\int_{t_2}^{T_j} dx' M(t_0,x,x') + \sqrt{\frac{\pi}{2}}B(t_0,t_1)C_2\sum_{i=1}^{N1} J_i\int_{t_0}^{t_2} dx\int_{t_1}^{T_i} dx' M(t_0,x,x') \\
&+ \sqrt{\frac{\pi}{2}}\frac{B(t_0,t_1)B(t_0,t_2)}{\sqrt{A_1}}\sum_{ii'=1}^{N1}\sum_{j=1}^{N2} G_{ii'}J_iJ_{i'}J_j\int_{t_1}^{T_i} dx\int_{t_2}^{T_j} dx' M(t_0,x,x') \\
&+ \sqrt{\frac{\pi}{2}}\frac{B(t_0,t_1)B(t_0,t_2)}{\sqrt{A_2}}\sum_{i=1}^{N1}\sum_{jj'=1}^{N2} J_iG_{jj'}J_jJ_{j'}\int_{t_1}^{T_i} dx\int_{t_2}^{T_j} dx' M(t_0,x,x') \\
&+ \frac{\pi}{2}B(t_0,t_1)B(t_0,t_2)\sum_{i=1}^{N1}\sum_{j=1}^{N2} J_iJ_j\int_{t_1}^{T_i} dx\int_{t_2}^{T_j} dx' M(t_0,x,x'),\end{aligned} \tag{16}$$

where A_1 and A_2 denote A for the two swaptions, respectively.

C. Market correlator

The forward bond price correlator G_{ij}, the swaption correlator, and volatility are all computed from a set of three-dimensional integrations on $M(t,x,x')$ with various integration limits. A general form of all the integration is given as follows:

$$\mathcal{I} = \int_{t_0}^{m1} dt\int_{m2}^{d1} dx\int_{m3}^{d2} dx' M(t,x,x') \tag{17}$$

and the limits of integrations are listed in Table I.

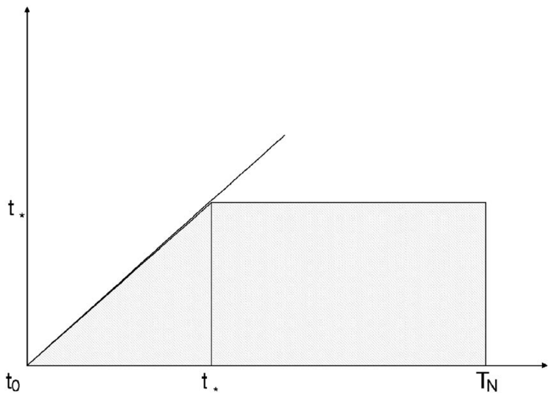

FIG. 1. The shaded area is the domain for evaluating the price of a swaption. For 2×10 swaption $t_*=t_0+2$ year and $T_N=t_*+10$ year.

Note that for quantities appearing in swaption price and volatility function, the swaption maturity is at t_* and the two indices i and j run from 1 to N, with the last payment being made at T_N. For the swaption correlation, options mature at two different times $t_2 \geq t_1$, and hence two indices i,j have the range $i=1,2,\ldots,N1$ and $j=a,a+1,\ldots,N2$ where the last payments are made at T_{N1} and T_{N2}, respectively. In the next section, we examine the data in detail in order to compute $\mathcal{I}$.

III. DATA FROM SWAPTION MARKET

The swaption market provides daily data for X by Y swaptions. These swaptions mature X years from today, with the underlying swap starting at time X and the last payment being paid $X+Y$ years in the future. The domain for the swaption instrument is given in the time and future time tx plane in Fig. 1.

All the prices are presented with interest rates in basis points (100 basis points=1% annual interest rate) and must be multiplied by the notional value of one million dollars. Daily swaption prices at the money are quoted from January 29, 2003 to January 28, 2005, a total of 523 daily data. In order to get accurate results, actual days in the real 6 months are divided by 360, since the convention for total number of days in a year is 360.

ZCYC data. In order to generate swaption prices and swaption correlation from the model, both the historical and current underlying forward interest rates are required. The value of the coupon bond and forward bond price and the par fixed rate R_P are computed from the current forward interest rates. The integrand of the forward bond correlator G_{ij}, namely $M(t,x,x')$, is derived from historical forward interest rates' data.

Our analysis uses Bloomberg data for the zero coupon yield curve (ZCYC), denoted by $Z(t_0,T)$, from January 29, 2003 to January 28, 2005, and which yields, in total, 523 daily ZCYC data. The ZCYC is necessary for evaluating long duration swaptions since Libor data exist for maturity of only up to a maximum of 7 years in the future, whereas ZCYC has data with maturity of up to 30 years.

The ZCYC is given in the $\theta=x-t=$constant direction as shown in Fig. 2, with the interval of θ between two data points not being a constant. Cubic spline is used for interpolating the data to a 3 month interval.

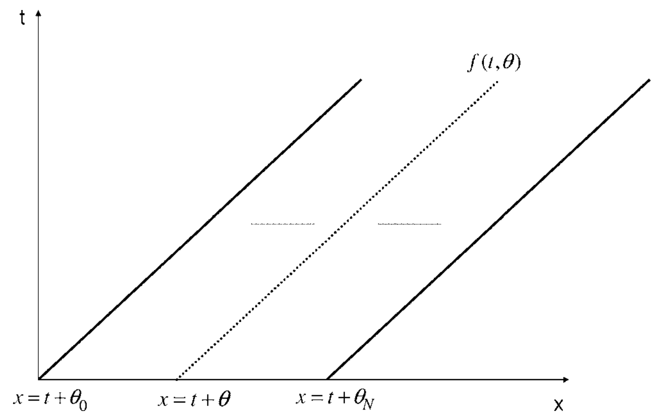

FIG. 2. Zero coupon yield curve data on lines of constant θ; the θ interval is 3 months by cubic spline. $\theta_N=30$ years.

From Ref. [5] we have that the zero coupon bond is given by

$$B(t_0,T) = \frac{1}{[1+Z(t_0,T)/c]^{(T-t_0)*c}}, \tag{18}$$

where c represents how many times the bond is compounded per year. For ZCYC c is given as half yearly, and hence we have $c=2/\text{year}$. As expected the forward bond price is given by

$$F(t_0,t_*,T) = \frac{B(t_0,T)}{B(t_0,t_*)}. \tag{19}$$

From the definition of the zero coupon bond,

$$B(t_0,T) = \exp\left(-\int_{t_0}^{T} dx f(t_0,x)\right),$$

we obtain, from Eq. (18), the following:

$$\int_{t_0}^{T} f(t_0,x)dx = \ln\{[1+Z(t_0,T)/c]^{(T-t_0)*c}\}. \tag{20}$$

Note the important fact that the bond market directly provides the ZCYC, which is the *integral* of the forward interest rates over future time x. One can numerically differentiate the ZCYC to extract $f(t,x)$; this procedure does yield an estimate of $f(t,x)$, but with such large errors that it makes the estimate quite useless for empirically analyzing swaption pricing. Hence we develop numerical procedures directly based on the ZCYC.

All data required for calculating a swaption's price can be obtained directly from the ZCYC data. The interpolation of ZCYC data and the convention used by Bloomberg have been empirically tested by comparing the computed R_P [from Eq. (3)] with the one given by market, and the result confirms the correctness of our computation.

IV. NUMERICAL ALGORITHM FOR THE FORWARD BOND CORRELATOR

The market value of the forward bond price correlator $\mathcal{I}$ given in Eq. (14) can be derived from ZCYC data. From Eq. (14) and for discrete time $\dot{f} \simeq \delta f / \epsilon$ the correlation for changes in the forward interest rates is given by [2]

$$M(t,x,x') = \frac{1}{\epsilon}\langle \delta f(t,x)\delta f(t,x')\rangle_c,$$

$$\delta f(t,x) = f(t+\epsilon,x) - f(t,x). \tag{21}$$

Thus, we have for the forward bond correlators the following:

$$\mathcal{I} = \frac{1}{\epsilon}\int_{t_0}^{m1} dt \int_{m2}^{d1} dx \int_{m3}^{d2} dx' \langle \delta f(t,x)\delta f(t,x')\rangle_c. \tag{22}$$

From Table I we see that none of the limits on the integrations over x,x' depend on the time variable t; hence the finite time difference operator δ can be moved out of the x,x' integrations and yields

$$\mathcal{I} = \frac{1}{\epsilon}\int_{t_0}^{m1} dt \left\langle \left(\delta \int_{m2}^{d1} dx f(t,x)\right)\left(\delta \int_{m3}^{d2} dx' f(t,x')\right)\right\rangle_c. \tag{23}$$

We keep to the x and x' integration variables instead of changing them to θ and θ' since, as discussed earlier, ZCYC data directly yield the integrals of forward interest rates on future time x. The numerical values of $\int_{m2}^{d1} dx f(t,x)$ and $\int_{m3}^{d2} dx' f(t,x')$ are obtained from the market values of the ZCYC.

To evaluate the market correlator $\mathcal{I}$ one needs to know the value of the correlator $M(t,x,x')$ in the future; the reason being that the time integration t in $\mathcal{I}$ runs from present time t_0 to time $m1 > t_0$ in the future. The problem of obtaining the future values of $M(t,x,x')$ can be solved by assuming that the correlation function for changes in the forward interest rates is invariant under time translations; that is

$$M(t,x,x') = M(t-a,x-a,x'-a). \tag{24}$$

The assumption of time translation invariance of the forward rates correlation function has been empirically tested in Ref. [6]; although this assumption cannot be indefinitely extended, a 2 year shift is considered to be reasonable [6].

The integration on the t axis can be converted to a summation by discretizing time into a lattice with spacing ϵ'; one then obtains

$$\mathcal{I} = \epsilon' \sum_{t_k=0}^{m1-t_0} \int_{m2}^{d1} dx \int_{m3}^{d2} dx' M(t_0+t_k,x,x'). \tag{25}$$

From Eq. (25) and a change of variables yields

$$x = y + t_k, \quad x' = y' + t_k.$$

We hence have from Eq. (25),

$$\mathcal{I} = \epsilon' \sum_{t_k} \int_{m2-t_k}^{d1-t_k} dy \int_{m3-t_k}^{d2-t_k} dy' M(t_0+t_k,y+t_k,y'+t_k)$$
$$= \epsilon' \sum_{t_k} \int_{m2-t_k}^{d1-t_k} dy \int_{m3-t_k}^{d2-t_k} dy' M(t_0,y,y'), \tag{26}$$

where the condition given in Eq. (24) has been used to obtain Eq. (26). The integration on future data has been replaced by a summation on the current value of $M(t_0,x,x')$, with x,x' taking values on various intervals. The current value of $M(t_0,x,x')$ in turn is evaluated by taking averages of the correlator over its past values.

From above and Eqs. (21), (23), and (26) we have

$$\mathcal{I} = \frac{\epsilon'}{\epsilon}\sum_{t_k} \left\langle \int_{m2-t_k}^{d1-t_k} \delta f(t_0,y)dy \int_{m3-t_k}^{d2-t_k} \delta f(t_0,y')dy' \right\rangle_c$$
$$= \frac{\epsilon'}{\epsilon}\sum_{t_k} \left\langle \left(\delta \int_{m2-t_k}^{d1-t_k} f(t_0,y)dy\right)\left(\delta \int_{m3-t_k}^{d2-t_k} f(t_0,y')dy'\right)\right\rangle_c.$$

As discussed earlier, in order to directly use the ZCYC data the finite time difference operator δ is taken outside the future time integrations. Note ϵ' is the time integration interval and is equal to ϵ; for the time summation with daily intervals $\epsilon = \epsilon' = \frac{1}{260}$ (260 is the actual number of trading days in one year).

Reexpressing $\mathcal{I}$ in terms of the ZCYC data we obtain

$$\mathcal{I} = \frac{\epsilon'}{\epsilon}\sum_{t_k} \langle \delta Y(t_0,m2-t_k,d1-t_k)\delta Y(t_0,m3-t_k,d2-t_k)\rangle_c,$$

where, from Eq. (20), we have

$$Y(t_0,t_*,T) = \int_{t_*}^{T} f(t_0,x)dx = \int_{t_0}^{T} f(t_0,x)dx - \int_{t_0}^{t_*} f(t_0,x)dx$$
$$= \ln[(1+Z(t_0,T)/c)^{(T-t_0)*c}]$$
$$- \ln[(1+Z(t_0,t_*)/c)^{(t_*-t_0)*c}]. \tag{27}$$

The forward bond price correlator's present value (at time t_0) is obtained by averaging the value of the correlator $\langle \delta Y(t_0,m2-t_k,d1-t_k)\delta Y(t_0,m3-t_k,d2-t_k)\rangle$ over the last $t_0 - t_A$ days with $t_A = 180$ days.[3] Since the computation requires the value of δY for different future time intervals x,x' we must again use cubic splines to interpolate ZCYC for obtaining daily values of the ZCYC. The shift of the future time integration to the present and the domain used for doing the averages for the correlator is illustrated in Fig. 3.

V. EMPIRICAL RESULTS

The 2×10 and 5×10 swaptions are priced for time series April 6, 2004–January 28, 2005 using the pricing formula from Sec. II. When computing the forward interest rates'

[3]We ran the program by adding 30 days to the time averaging for evaluating the expectation values of the correlators; the best fit is given when the averaging is done over the past 180 days; see Sec. V.

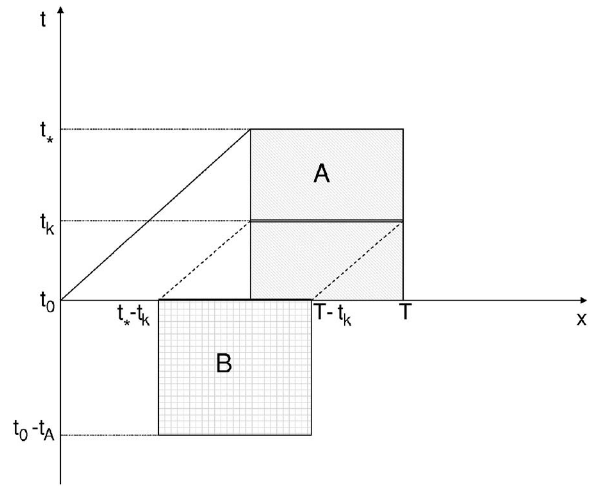

FIG. 3. Shaded area A is the integration domain of $\mathcal{I}$. For the case when $t=t_0+t_k$, the integration of x and x' for evaluating the expression for $Y(t_0+t_k,t_*,T)$ inside $\langle \delta Y(t_0+t_k,t_*,T)\delta Y(t_0+t_k,t_*,T)\rangle$ is shifted back to t_0. Invariance in time yields this to be equal to $\langle \delta Y(t_0,t_*-t_k,T-t_k)\delta Y(t_0,t_*-t_k,T-t_k)\rangle$. A historical average is done over the shaded area B, which is in the past of t_0. $t_A=180$ days is the optimum number of past data for evaluating the historical averages.

correlator $M(t;x,x')$ we found that daily swaption prices are stable when more than 270 days of historical data for ZCYC are used; but a 270-day average does not give the best fit of the predictions of model swaption price with the swaption's market value. This may be due to too much old information creating large errors in the predictions for the present day swaption prices. However, averaging on less historical data causes the swaption price curve to fluctuate strongly since it is likely that new information dominates swaption pricing and makes the price too sensitive to small changes.

Our empirical studies and results show that a moving averaging of 180 days of historical data gives the best result for this period. One can most likely improve the accuracy by higher frequency sampling of 180 days of historical data.

The results obtained from the field theory model are compared with daily market data and are shown in Figs. 4 and 5, with normalized root mean square of error being 3.31% and 6.31%, respectively.

The results for the swaption volatility and correlation discussed in Sec. II B are derived for the change on the same instruments; from Eq. (22)

TABLE I. The various domains of integration for evaluating the integral $\mathcal{I}=\int_{t_0}^{m1}dt\int_{m2}^{d1}dx\int_{m3}^{d2}dx' M(t,x,x')$ that are required for computing the coefficients in the swaption price and correlators.

	$m1$	$m2$	$m3$	$d1$	$d2$
G_{ij}	t_*	t_*	t_*	T_i	T_j
$G_{ii'}$	t_1	t_1	t_1	T_i	$T_{i'}$
$G_{jj'}$	t_2	t_2	t_2	T_j	$T_{j'}$

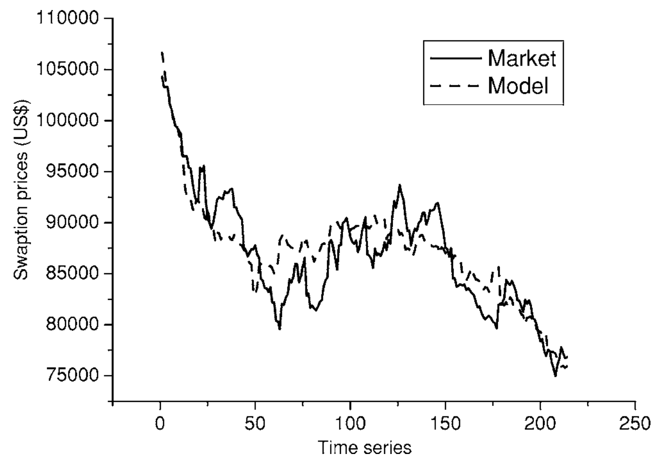

FIG. 4. 2×10 swaption price versus time t_0 (April 6, 2004–January 28, 2005), for both market and model. Normalized root mean square error=3.31%.

$$\delta C_1 \equiv C_1(t_0+\epsilon) - C_1(t_0) \equiv C_1(t_0+\epsilon,R_S) - C_1(t_0,R_S), \tag{28}$$

where $C_1(t_0+\epsilon)$ and $C_1(t_0)$ are the same contract being traded on successive days. Par fixed rate R_P is determined when the contract is initiated at time t_0, and the swaption $C_1(t_0)$ is at the money. However, in general $C_1(t_0+\epsilon)$ is away from the money; the reason being that the swaption depends on the forward bond prices F_i, and these change every day and hence there is a daily change in the par fixed rate R_p. From the market we only have the price of the swaption at the money. Historical data for the daily prices of swaptions in the money and out of the money are not quoted by Bloomberg. Hence, only the swaption volatility and correlation computed from the model are shown in Fig. 6, without any comparison made with the market value for these quantities.

Comparison of field theory pricing with HJM model. In order to see how the field theory model compares with the industry standard one factor HJM model, we empirically

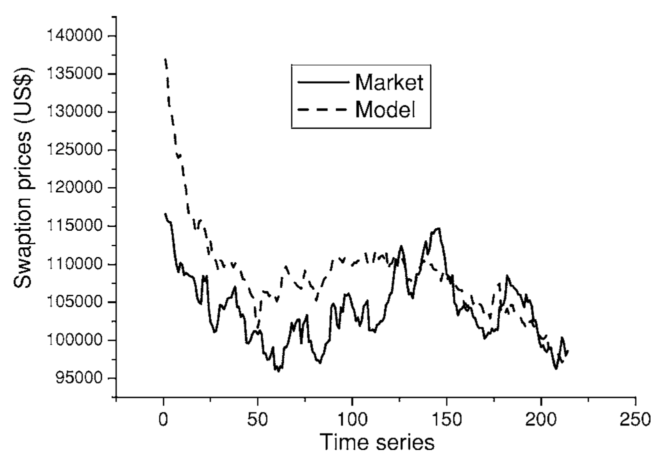

FIG. 5. 5×10 swaption price versus time t_0 (April 6, 2004–January 28, 2005), both market and model. The normalized root mean square error=6.31%.

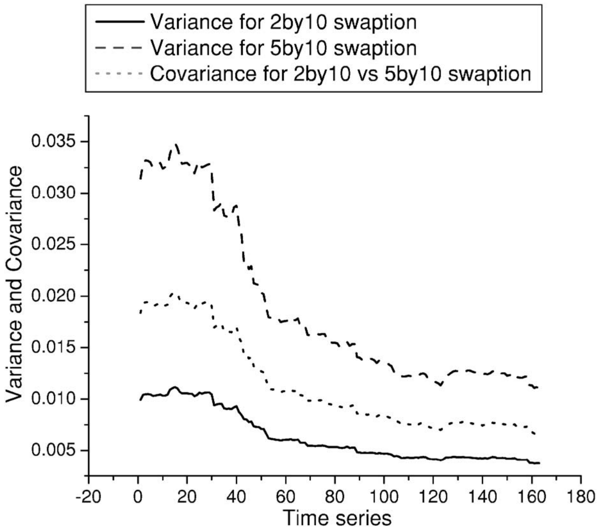

FIG. 6. Swaption variance $\langle \dot{C}_1^2 \rangle_c$, $\langle \dot{C}_2^2 \rangle_c$ and covariance $\langle \dot{C}_1 \dot{C}_2 \rangle_c$ versus time t_0 (June 15, 2004–January 27, 2005) computed from the quantum finance model, with the value of the forward bond prices taken from market data.

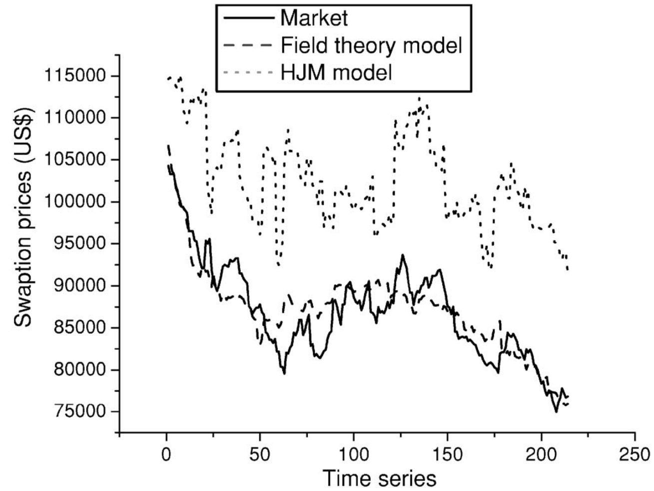

FIG. 7. 2×10 swaption price, at the money, from the market, from the quantum finance model, and from the HJM model. Time t_0 is in the range (April 6, 2004–January 28, 2005). The normalized root mean square error for HJM=18.87% compared with the far more accurate quantum finance swaption formula with error=3.31%.

study swaption pricing in the HJM model. By considering the volatility function to have the special form of $\sigma(t,x) = \sigma_0 e^{-\lambda(x-t)}$ Jarrow and Turnbull [5] obtained, for the one-factor HJM model, the following explicit expression for the coupon bond option:

$$C_{\mathrm{HJM}}(t_0,t_*,K) = \sum_{i=1}^{N} c_i B(t_0,T_i) N(d_i) - K B(t_0,t_*) N(d),$$

$$d_i \equiv \frac{r'}{\sigma_R} + W(t_*,T_i)\sigma_R, \quad d = \frac{r'}{\sigma_R},$$

$$W(t_*,T_i) \equiv \frac{1}{\lambda}(1 - e^{-\lambda(T_i - t_*)}), \quad \sigma_R^2 = \frac{\sigma_0^2}{2\lambda}(1 - e^{-2\lambda(t_* - t_0)}). \tag{29}$$

The quantity r' is related to the strike price K by a nonlinear transformation that depends on the initial coupon bond price [5]. As shown in paper I, to leading order in σ_0 the HJM limit of the field theory pricing formula with exponential volatility given by $\sigma(t,x) = \sigma_0 \exp[-\lambda(x-t)]$ yields the HJM pricing formula.

We estimate σ_0 for the exponential volatility function in the HJM model from historical ZCYC data. By using exponential volatility and daily forward bond prices obtained from ZCYC, we price the swaption with the HJM pricing formula and in Fig. 7 compare it with the market price and the field theory price.

The results show that the HJM model is inadequate for pricing swaptions, both because it systematically overprices the swaption by a large amount, and also because the instability of the price itself would give incorrect results if one tries to hedge the swaption using the one-factor HJM pricing formula.

Instead of using the HJM formula for pricing the coupon bond options practitioners may consider representing the price of the swaption by an implied volatility using the HJM pricing formula. However, unlike the case for the price of caps where this procedure is possible [3], the entire swaption curve cannot be fitted by adjusting only one quantity σ_0. Furthermore, the implied volatility $\sigma(t,x)$ in the first place may not be able to fit the price of all swaptions, and second, it will depend on time; it is quite impractical to numerically evaluate daily implied volatility from daily swaption prices.

VI. CONCLUSION

The quantum finance swaption pricing formula was empirically tested, for various durations, by comparing its predictions with the market values. There is over 90% agreement of the theoretical predictions for the swaption's price with its market value, with errors around 6% for most swaptions and with an accuracy of about 3% for the shorter maturity swaptions.

A comparison of the field theory model and the industry standard HJM model shows that the field theory model gives a more accurate and stable result than the HJM model.

The HJM model is not suited for pricing swaptions because the volatility parameter that goes into the pricing formula cannot be extracted from the swaption data. In contrast since the field theory model directly uses the market correlator $M(t,x,x')$ all the market information is fully accounted for in the swaption price.

The correlation of different swaptions and their volatilities are central ingredients in forming swaption portfolios and hedging these portfolios. The quantum finance swaption pricing formula provides an approximate analytic result that can in turn be used to compute the correlation and volatility of swaptions; based on these analytic results one can form and hedge interest rate portfolios.

ACKNOWLEDGMENTS

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APPENDIX: TEST OF ALGORITHM FOR COMPUTING $\mathcal{I}$

The computation of $\mathcal{I}$ is the key step in calculating swaption prices. We test the program used for numerically computing $\mathcal{I}$ by an analytically solvable formula of the forward interest rates.

Consider an analytical formula for forward interest rates as given below

$$f(t,x) = 1 - e^{-\lambda(x-t)}. \tag{A1}$$

The forward interest rates have an exponential form and increase from $f(t,t)=0$ to the maximum value 1. Furthermore, $f(t,x)$ depends only on $x-t$, which is what we need for carrying out the shift of time as explained in Eq. (24).

Since we can analytically perform the integration of forward interest rates, one can directly determine Y. The analytic expression for the Y is given by

$$\begin{aligned} Y &\equiv Y(t,t_*,T_i) \\ &= \int_{t_*}^{T_i} dx f(t,x) \\ &= T_i - t_* + \frac{1}{\lambda} e^{\lambda t}(e^{-\lambda T_i} - e^{-\lambda t_*}). \end{aligned} \tag{A2}$$

The input data are generated from our test forward interest rates and processed using the same algorithm as employed in Sec. IV. Using the forward interest rates itself as input data will cause new errors since it does not directly appear in the program being checked. Note that Y depends on three variables and is not suitable as input data. In analogy with Eq. (27) we form a new variable $z(t,x)$, similar to ZCYC, such that

$$\begin{aligned} Y(t,t_*,T_i) &= z(t,T_i) - z(t,t_*), \\ z(t,x) &= x - t + \frac{1}{\lambda}(e^{-(x-t)} - 1). \end{aligned} \tag{A3}$$

The function $z(t,x)$ from the above formula is used as input data since this is a starting point for the analysis of market data.

Since we are checking $\mathcal{I}$, more concretely G_{ij}, we have the exact analytical result

$$G_{ij} = \int_{t_0}^{t_*} dt \langle \dot{Y}(t,t_*,T_i)\dot{Y}(t,t_*,T_j)\rangle_c \tag{A4}$$

and from Eq. (A2) we have

$$\dot{Y}(t,t_*,T_i) = e^{\lambda t}(e^{-\lambda T_i} - e^{-\lambda t_*}) \equiv b_i e^{\lambda t}. \tag{A5}$$

Changing the integration on t to a summation, we have

$$G_{ij} = \epsilon' \sum_{k=0}^{N} \int_{t_*}^{T_i} dx \int_{t_*}^{T_j} dx' M(x-(t_0+k\epsilon'), x'-(t_0+k\epsilon')), \tag{A6}$$

where $N=(t_*-t_0)/\epsilon'$. Note

$$\begin{aligned} &\epsilon' \sum_{k=0}^{N} \int_{t_*-k\epsilon'}^{T_i-k\epsilon'} dy \int_{t_*-k\epsilon'}^{T_j-k\epsilon'} dy' M(y-t_0, y'-t_0) \\ &= \epsilon' \sum_{k=0}^{N} \langle \dot{Y}(t_0,t_*-k\epsilon',T_i-k\epsilon') \\ &\quad \times \dot{Y}(t_0,t_*-k\epsilon',T_j-k\epsilon')\rangle_c, \end{aligned}$$

and

$$\begin{aligned} &\langle \dot{Y}(t_0,t_*-k\epsilon',T_i-k\epsilon')\dot{Y}(t_0,t_*-k\epsilon',T_j-k\epsilon')\rangle_c \\ &= \frac{1}{N'} \sum_{n=0}^{N'-1} \dot{Y}(t_0-n\epsilon,t_*-k\epsilon',T_i-k\epsilon')\dot{Y}(t_0-n\epsilon,t_* \\ &\quad -k\epsilon',T_j-k\epsilon') - \frac{1}{N'^2}\sum_{n=0}^{N'-1} \dot{Y}(t_0-n\epsilon,t_*-k\epsilon',T_i-k\epsilon') \\ &\quad \times \sum_{n=0}^{N'-1} \dot{Y}(t_0-n\epsilon,t_*-k\epsilon',T_j-k\epsilon') \\ &= \frac{1}{N'} b_i b_j \frac{(1-e^{-2\lambda\epsilon N'})}{1-e^{-2\lambda\epsilon}} - \frac{1}{N'^2} b_i b_j \frac{(1-e^{-\lambda\epsilon N'})^2}{(1-e^{-\lambda\epsilon})^2} \end{aligned} \tag{A7}$$

with

$$b_i = e^{-\lambda(T_i-k\epsilon')} - e^{-\lambda(t_*-k\epsilon')}.$$

We ran the program for $\mathcal{I}$ with our artificial test input and compared the result from the program with the known analytical results. The numerical test for the algorithm shows that it exactly reproduces the analytical results, verifying the correctness of the algorithm used for our empirical analysis of swaptions.

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